POLISHNESS OF THE WIJSMAN TOPOLOGY REVISITED

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Abstract. Let $X$ be a completely metrizable space. Then the space of nonempty closed subsets of $X$ endowed with the Wijsman topology is $\alpha$-favorable in the strong Choquet game. As a consequence, a short proof of the Beer-Costantini Theorem on Polishness of the Wijsman topology is given.

Denote by $CL(X)$ the nonempty closed subsets of the metric space $(X,d)$. The Wijsman topology $\tau_W$ on $CL(X)$ is the weak topology generated by $\{d(x,\cdot) : x \in X\}$, where the distance functional $d(x,A) = \inf\{d(x,a) : a \in A\}$ is viewed as a function of the set argument $A \in CL(X)$. It was shown by G. Beer in [Be1] (see also [Be2]) that given a separable complete metric space $(X,d)$, the corresponding hyperspace $(CL(X),\tau_W)$ is Polish. Since even uniformly equivalent metrics on $X$ may give rise to different Wijsman topologies (cf. [LL],[CLZ]), it required a separate argument to show that Polish base spaces always generate Polish Wijsman topologies. This was established by C. Costantini in [Co1]. Thus, combining these results and the fact that $X$ embeds in $(CL(X),\tau_W)$ as a closed subspace, one gets

Theorem 1. The space $(CL(X),\tau_W)$ is Polish if and only if $(X,d)$ is Polish.

It is the purpose of this note to present a short proof of the above theorem based on the so-called strong Choquet game (cf. [Ch] or [Ke]). In the game, denoted by $\Gamma$, two players $\alpha$ and $\beta$ take turns in choosing objects in the topological space $X$ with an open base $B$: $\beta$ starts by picking $(x_0,V_0)$ from $E(X,B) = \{(x,V) \in X \times B : x \in V\}$ and $\alpha$ responds by $U_0 \in B$ with $x_0 \in U_0 \subset V_0$. The next choice of $\beta$ is some couple $(x_1,V_1) \in E(X,B)$ with $V_1 \subset U_0$ and again $\alpha$ picks $U_1$ with $x_1 \in U_1 \subset V_1$ etc. Player $\alpha$ wins the run $(x_0,V_0),U_0,\ldots,(x_n,V_n),U_n,\ldots$ provided $\bigcap_n U_n = \bigcap_n V_n \neq \emptyset$, otherwise $\beta$ wins. A winning tactic (abbr. w.t.) for $\alpha$ (cf. [Ch]) is a function $\sigma : E(X,B) \to B$ such that $\alpha$ wins every run of $\Gamma$ compatible with $\sigma$, i.e. such that $U_n = \sigma(x_n,V_n)$ for all $n$. The game $\Gamma$ is $\alpha$-favorable if $\alpha$ possesses a winning tactic; in this case $X$ is called a strong Choquet space (see [Ke]). The Choquet Theorem (see [Ch], Theorem 8.7 or [Ke], Theorem 8.17) claims that a metrizable space is completely metrizable if and only if it is a strong Choquet space.

In the sequel $\omega$ will stand for the nonnegative integers, $B(x,\varepsilon)$ for the closed ball about $x \in X$ of radius $\varepsilon$ in the metric space $(X,d)$ and $B^c$ for the complement of $B \subset X$, respectively. For $U \subset X$ put $U^- = \{A \in CL(X) : A \cap U \neq \emptyset\}$. It is a

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routine to show that a base $B_W$ for the Wijsman topology consists of the sets of the form

$$(V_0, \ldots, V_k)_D = \bigcap_{i \leq k} V_i^+ \cap \bigcap_{j \leq m} \{A \in CL(X) : d(x_j, A) > \varepsilon_j\}$$

with $V_i \subset X$ open ($i \leq k$), $x_j \in X$, $\varepsilon_j > 0$ ($j \leq m$), $D = (x_0, \ldots, x_m; \varepsilon_0, \ldots, \varepsilon_m)$. For every $D$ of this kind, denote $M(D) = \bigcup_{j \leq m} B(x_j, \varepsilon_j)$.

Since metrizability of the Wijsman topology is equivalent to separability of the base space $X$ (see [Be2], Theorem 2.1.5), in order to prove Theorem 1 it suffices by the Choquet Theorem to prove

**Theorem 2.** If $X$ is completely metrizable, then $(CL(X), \tau_W)$ is a strong Choquet space.

**Proof.** Assume that $(X, \tau)$ is completely metrizable and $d$ is a compatible metric on $X$. Then by the Choquet Theorem we can find a w.t. $\sigma : \mathcal{E}(X, \tau) \to \tau$ for $\alpha$. Define a tactic $\sigma_W : \mathcal{E}(CL(X), B_W) \to B_W$ for $\alpha$ as follows: first, for each $V \in \tau$ and $A \in V^-$ fix a point $x_{A,V} \in A \cap V$. Then given $(A, V) \in \mathcal{E}(CL(X), B_W)$ with $V = (V_0, \ldots, V_k)_D$ and $D = (x_0, \ldots, x_m; \varepsilon_0, \ldots, \varepsilon_m)$ define

$$\sigma_W(A, V) = (\sigma(x_{A,V_0} \cap M(D)^c), V_0 \cap M(D)^c), \ldots, \sigma(x_{A,V_k} \cap M(D)^c), V_k \cap M(D)^c)_{\hat{D}}$$

where $\hat{D} = (x_0, \ldots, x_m; \tilde{\varepsilon}_0, \ldots, \tilde{\varepsilon}_m)$ with $\tilde{\varepsilon}_j = \frac{\varepsilon_j + d(x_j, A)}{2}$ for all $j \leq m$. Then $A \in \sigma_W(A, V) \subset V$. We will show that $\sigma_W$ is a winning tactic for $\alpha$.

Indeed, suppose that $(A_0, V_0), U_0, (A_n, V_n), U_n, \ldots$ is a run of $\Gamma$ in $CL(X)$ such that $U_n = \sigma_W(A_n, V_n)$ for all $n$. Denote $U_n = (U^n_0, \ldots, U^n_{l_n})_{B_n}$ and $V_n = (V^n_0, \ldots, V^n_{k_n})_{D_n}$ for appropriate $B_n$ and $D_n$. Observe that $V_{n+1} \subset U_n$ and $V_{n+1} \neq \emptyset$ implies that $M(D_{n+1})^c \subset M(B_n)^c$ and for all $s \leq l_n$ there exists $t \leq n_{n+1}$ such that $V_{t+1} \cap M(D_{n+1})^c \subset U^n_s \cap M(B_n)^c$. Hence, without loss of generality, assume that $k_{n+1} > l_n = k_n$ and $t = s$. Put $l_{n-1} = -1$. Then for all $n \in \omega$ and $l_{n-1} < i \leq l_n$,

$$(x_{A_n}, V_i^n \cap M(D_n)^c, V_i^n \cap M(D_n)^c), U_i^n \cap M(B_n)^c, \ldots,$$

$$\quad (x_{A_{n+j}}, V_i^{n+j} \cap M(D_{n+j})^c, V_i^{n+j} \cap M(D_{n+j})^c), U_i^{n+j} \cap M(B_{n+j})^c, \ldots$$

is a run of $\Gamma$ in $X$ compatible with $\sigma$, so there exists $a_i \in \bigcap_{j \in \omega} U_i^{n+j} \cap M(B_{n+j})^c$. Denote by $A$ the closure of $\{a_i : i \in \omega\}$ in $X$. Fix $n$ and $i$. Then for some $N > n$, $a_i \in U_i^N \cap M(B_N)^c \subset M(B_{n+1})^c = M(D_n)^c$.

Consequently, if $D_n = (x_0, \ldots, x_m; \varepsilon_0, \ldots, \varepsilon_m)$ then for all $j \leq m$, $d(x_j, a_i) \geq \tilde{\varepsilon}_j$, thus $d(x_j, A) = \inf\{d(x_j, a_i) : i \in \omega\} \geq \tilde{\varepsilon}_j > \varepsilon_j$. It follows now that $A \in V_n$ for all $n \in \omega$, whence $\alpha$ wins the run. □

**Remark.** It is known that metrizable spaces that are not $\beta$-favorable in $\Gamma$ are the hereditarily Baire spaces (see [De]), i.e. spaces, every closed subspace of which is a Baire space. In general however even strong Choquet spaces may be non-hereditarily Baire ([De]). It could be of interest therefore to find out if complete metrizability of $X$ implies hereditary Baireness of $(CL(X), \tau_W)$. Note that it will be certainly a Baire space ([Zs]) and it could be non-Čech-complete, as was shown in [Co2].
References


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