ON DENSITY OF RATIO SETS OF POWERS OF PRIMES

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Introduction

Denote by $\mathbb{R}^+$ and $\mathbb{N}$ the set of all positive real numbers and the natural numbers, respectively. Let $P = \{p_1, \ldots, p_n, \ldots\}$ be the set of all primes enumerated in increasing order. Let $P = \{p_1, \ldots, p_n, \ldots\}$ be the set of all primes enumerated in increasing order. Denote by $R(A, B) = \{a/b; a \in A, b \in B\}$ the ratio set of $A, B \subset \mathbb{R}^+$ and put $R(A) = R(A, A)$ for $A \subset \mathbb{R}^+$ (cf. [3],[4],[5]). Note that $R(A, B) \neq R(B, A)$ in general, however $R(A, B)$ is dense in $\mathbb{R}^+$ if and only if $R(B, A)$ is dense in $\mathbb{R}^+$.

If we consider the sets of powers of prime numbers $K_\alpha = \{p^\alpha; p \in P\}$ for $\alpha > 0$ and $K = \{p^\alpha; p \in P\}$ it can be shown that they are sparser on the real line than $P$. More precisely, denote by $K_\alpha(x), K(x)$ and $\pi(x)$, respectively, the number of elements of $K_\alpha$, $K$ and $P$, respectively, not exceeding $x \in \mathbb{N}$. Then clearly $K_\alpha(x) = \pi(x^{1/\alpha})$, hence if $\alpha > 1$, then

$$0 \leq K_\alpha(x) = \frac{\pi(x^{1/\alpha})}{\pi(x)} \sim \alpha x^{1/\alpha-1} \rightarrow 0 \ (x \to \infty)$$

by the Prime Number Theorem (cf.[2],p.152). Further let $\alpha > 0$. Choose $\beta > \alpha$ and let $x$ be sufficiently large so that $p_n > \beta$, where $n = K(x)$. Then $K(x) \leq K_\beta(x)$ and hence

$$0 \leq K(x) \frac{K_\beta(x)}{K_\alpha(x)} = \frac{\pi(x^{1/\beta})}{\pi(x^{1/\alpha})} \sim \frac{\beta}{\alpha} x^{1/\beta-1/\alpha} \rightarrow 0 \ (x \to \infty)$$

by the Prime Number Theorem.

On the other hand it is known that $R(P)$ is dense in $\mathbb{R}^+$ (see [2],p.155 and [1]), thus in light of the previous considerations it is of interest to investigate what is the density of sets $R(A, B)$ if we choose for $A, B$ the sets $K_\alpha$ and $K$, respectively.

It is the purpose of this note to show that the sets $R(K_\alpha, K_\beta)$ and $R(K_\alpha, K)$, respectively are still dense in $\mathbb{R}^+$, while $R(K)$ already consists of isolated points in $\mathbb{R}^+$ as do the larger sets $A_0 = \{(\frac{m}{n})^m; m, n \in \mathbb{N}\}$ and $A_1 = \{(\frac{m}{n})^n; m, n \in \mathbb{N}\}$, respectively.

We will say that the set $A$ of values of the sequence $\{a_n\}_{n=1}^\infty$ of positive real numbers is lacunary if $\liminf_{n \to \infty} \frac{a_{n+1}}{a_n} > 1$. The symbol $X^d$ will stand for the set of all accumulation points of $X \subset \mathbb{R}^+$. 

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Main Results

An argument similar to that of [2] justifying the density of \( R(P) \) in \( \mathbb{R}^+ \) yields the following

**Proposition 1.** The set \( R(K\alpha,K\beta) \) is dense in \( \mathbb{R}^+ \) for every \( \alpha, \beta > 0 \).

**Proof.** Choose \( 0 < a < b \) arbitrarily. It is not hard to show by the Prime Number Theorem ([2], p.152) that \( \frac{\pi((bx)^{1/\alpha})}{\pi((ax)^{1/\alpha})} \to \frac{b}{a}^{1/\alpha} \) as \( x \to \infty \). Further \( \frac{b}{a}^{1/\alpha} > 1 \), and consequently there exists an \( x_0 > 0 \) such that \( \pi((bx)^{1/\alpha}) - \pi((ax)^{1/\alpha}) > 0 \) for all \( x \geq x_0 \).

It means that there exist primes \( p, q \) such that
\[
(aq^\beta)^{1/\alpha} < p < (bq^\beta)^{1/\alpha}, \quad \text{i.e.} \quad \frac{p^\alpha}{q^\beta} \in (a, b) \cap R(K\alpha,K\beta).
\]
Hence \( R(K\alpha,K\beta) \) is dense in \( \mathbb{R}^+ \). □

Further we have

**Proposition 2.** The set \( R(K,K\alpha) \) is dense in \( \mathbb{R}^+ \) for all \( \alpha > 0 \).

**Proof.** Choose \( 0 < a < b \) arbitrarily. It is known that \( \frac{p_{n+1}}{p_n} \to 1 \) as \( n \to \infty \) (see [2], p.153), thus \( \left( \frac{p_{n+1}}{p_n} \right)^\alpha \to 1 \) as \( n \to \infty \).

Therefore we can find \( m \in \mathbb{N} \) such that for each \( n \geq m \)

\[
(1) \quad \frac{p_{n+1}^\alpha}{p_n^\alpha} < \frac{b}{a}.
\]

Pick \( p_0 \in P \) for which \( p_0^\alpha a < p_0^\alpha \) and put
\[
q_0 = \max\{p \in P; p \geq p_m \text{ and } p^\alpha a < p_0^\alpha\}.
\]
Then clearly \( q_0 \in P \), say \( q_0 = p_s \) \((s \geq m)\), and by (1) we get
\[
q_0^\alpha a = p_s^\alpha a < p_0^\alpha \leq p_{s+1}^\alpha a < p_s^\alpha b = q_0^\alpha b, \quad \text{hence} \quad \frac{p_0^\alpha}{q_0^\beta} \in (a, b) \cap R(K,K\alpha),
\]
which proves the density of \( R(K,K\alpha) \) in \( \mathbb{R}^+ \). □

The following proposition demonstrates that \( R(K) \) cannot be dense:

**Proposition 3.** Let the sequence \( 0 < a_1 < a_2 < \ldots \) be lacunary. Denote
\[
\liminf_{n \to \infty} \frac{a_{n+1}}{a_n} = c \text{ where } c \in (1, +\infty].
\]
Then \( R(A) \) is not dense in \( \mathbb{R}^+ \), moreover
\[
(2) \quad R(A)^d \cap \left( \frac{1}{c^d}, c \right) = \emptyset.
\]
Proof. Let \( t \in R(A)_d \). Clearly \( t \neq 1 \). Suppose \( t > 1 \). Then there exist increasing sequences \( \{m_k\}_{k=1}^{\infty} \) and \( \{n_k\}_{k=1}^{\infty} \) of natural numbers such that \( m_k > n_k \) \((k \in \mathbb{N})\) and \( \frac{a_{n_k}}{a_{n_k}} \to t \) \((k \to \infty)\). The sequence \( \{a_n\}_{n=1}^{\infty} \) is increasing thus

\[
t \geq \liminf_{n \to \infty} \frac{a_{n+1}}{a_n} = c.
\]

If \( t < 1 \) we analogously get that

\[
t \leq \limsup_{n \to \infty} \frac{a_n}{a_{n+1}} = \frac{1}{c}.
\]

This justifies (2). \( \square \)

Corollary. Each point of \( R(K) \) is isolated in \( \mathbb{R}^+ \).

Proof. It suffices to observe that \( K \) is lacunary, moreover \( \lim_{n \to \infty} \frac{\frac{p_{n+1}}{p_n}}{\frac{1}{n}} = +\infty \).

Therefore in view of (2) we get \( R(K)_d \cap (0, +\infty) = \emptyset \). \( \square \)

Finally we have

Proposition 4. The sets \( A_0 \) and \( A_1 \) consist of isolated points in \( \mathbb{R}^+ \).

Proof. Let \( t \in A_0^d \). We can easily see that \( t \neq 1 \). Suppose \( t > 1 \) and denote by \( t_k = \left(\frac{m_k}{n_k}\right) = A_0 \) a sequence converging to \( t \) such that \( t_k > 1 \) for all \( k \in \mathbb{N} \). There are two possibilities:

i) there exists an \( s \in \mathbb{N} \) such that \( m_k = n_k + s \) for infinitely many indices \( k \).

Then we can find a subsequence \( \{t_{k_l}\}_{l=1}^{\infty} \) of \( \{t_k\}_{k=1}^{\infty} \) for which

\[
t_{k_l} = \left(\frac{n_{k_l} + s}{n_{k_l}}\right)^{n_{k_l} + s} = (1 + \frac{s}{n_{k_l}})^{n_{k_l}} (1 + \frac{s}{n_{k_l}})^s \to e^s,
\]

as \( l \to \infty \).

Consequently \( t = e^s \).

ii) for all \( s \in \mathbb{N} \) there are only finitely many \( k \)'s such that \( m_k = n_k + s \). Then there exists an increasing sequence \( \{k_s\}_{s=1}^{\infty} \) of natural numbers such that \( m_{k_s} > n_{k_s} + s \) for each \( s \in \mathbb{N} \). In view of the well-known Bernoulli’s inequality we get

\[
t_{k_s} = \left(\frac{m_{k_s}}{n_{k_s}}\right)^{m_{k_s}} > (1 + \frac{s}{n_{k_s}})^{n_{k_s}} \geq 1 + s
\]

for every \( s \in \mathbb{N} \),

accordingly \( t = +\infty \).

Considering the case \( t < 1 \) we can similarly get that \( e^{-s} \in A_0^d \) for all \( s \in \mathbb{N} \) and \( 0 \in A_0^d \). It is now clear that

\[
A_0^d = \{0, +\infty\} \cup \{e^s; s = \pm 1, \pm 2, \ldots\},
\]

thus \( A_0 \cap A_0^d = \emptyset \) since the elements of \( A_0 \) are positive rationals.

A similar argument applies to \( A_1 \). \( \square \)

Remark. In connection with Proposition 1 and the Corollary Prof. Šalát has posed the following problem:

Characterize sequences \( \{\alpha_n\}_{n=1}^{\infty} \) of positive reals for which the ratio set of \( \{p_1^{\alpha_1}, p_2^{\alpha_2}, \ldots\} \) is dense in \( \mathbb{R}^+ \).
References