ON SUBSPACES OF MEASURABLE REAL FUNCTIONS

LÁSZLÓ ZSILINSZKY

College of Education, Department of Mathematics
Farská 19, 949 74 Nitra, Slovakia

Abstract. Let \((X, S, \mu)\) be a measure space. Let \(\Phi : \mathbb{R} \rightarrow \mathbb{R}\) be a continuous function. Topological properties of the space of all measurable real functions \(f\) such that \(\Phi \circ f\) is Lebesgue-integrable are investigated in the space of measurable real functions endowed with the topology of convergence in measure.

Introduction

Let \((X, S, \mu)\) be a measure space. Denote by \(\mathcal{M}\) the space of all measurable real functions on \(X\). As usual the symbol \(L_p(\mu)\) stands for the set of all functions \(f \in \mathcal{M}\) for which \(\int_X |f|^p d\mu < +\infty \) \((p \geq 1)\).

It is shown in [4] that the Riemann-integrable functions on the interval \([a, b]\) \((a, b \in \mathbb{R})\) constitute a meager set in the space of all Lebesgue-integrable functions on \([a, b]\) furnished with the topology of mean convergence. Then a natural question arises to establish the largeness of Lebesgue-integrable functions, or more generally of \(L_p\) spaces in the space \(\mathcal{M}\) with an appropriate topology.

Making allowance for this we could pursue the analogy further by examining the class \(A(\Phi)\) of all measurable real functions \(f\) such that \(\Phi \circ f\) is Lebesgue-integrable, where \(\Phi : \mathbb{R} \rightarrow \mathbb{R}\) is an arbitrary but fixed continuous function.

In favour of this we need a proper topology on \(\mathcal{M}\). Let \(E(f, g; r) = \{x \in X; |f(x) - g(x)| > r\}\), where \(f, g \in \mathcal{M}, r > 0\). Define the pseudo-metric \(\varrho\) on \(\mathcal{M}\) as follows ([1]):

\[
\varrho(f, g) = \inf\{r > 0; \mu(E(f, g; r)) \leq r\} \quad (f, g \in \mathcal{M}).
\]

Given \(f_n, f \in \mathcal{M} \quad (n \in \mathbb{N})\) we say that \(f_n\) converges in measure to \(f\) if, for each \(r > 0\)

\[
\lim_{n \to \infty} \mu(E(f_n, f; r)) = 0.
\]

It is known that the \(\varrho\)-convergence is equivalent to the convergence in measure, further \((\mathcal{M}, \varrho)\) is a complete pseudo-metric space ([1],p.80).

Define the following sets:

\[
1991 \text{ Mathematics Subject Classification.} \quad 28A20, 54E52.
\]

\textit{Key words and phrases.} Convergence in measure, Baire category, \(L_p\) spaces.

Typeset by A\LaTeX
\[ A_\alpha(\Phi) = \{ f \in \mathcal{M} ; \int_X |\Phi \circ f|d\mu \leq \alpha \} \ (\alpha \geq 0), \]

\[ A(\Phi) = \{ f \in \mathcal{M} ; \int_X |\Phi \circ f|d\mu < +\infty \}, \]

where \( \Phi : \mathbb{R} \to \mathbb{R} \) is an arbitrary but fixed continuous function.

The symbol \( \chi_A \) stands for the characteristic function of \( A \subset X \).

**Main Results**

First we point out to which Borel class \( A_\alpha(\Phi) \) and \( A(\Phi) \), respectively belong \((\alpha \geq 0)\). We have

**Theorem 1.** The set \( A_\alpha(\Phi) \) is closed in \((\mathcal{M}, \varrho)\) for all \( \alpha \geq 0 \).

**Proof.** Let \( f \in \mathcal{M} \), \( f_n \in A_\alpha(\Phi) \) and \( \varrho(f_n, f) \to 0 \ (n \to \infty) \). Then by a well-known theorem of Riesz there exists a subsequence \( \{f_{n_k}\}_{k=1}^{\infty} \) of \( \{f_n\}_{n=1}^{\infty} \) converging a.e. on \( X \) to \( f \). Consequently \( |\Phi \circ f_{n_k}| \to |\Phi \circ f| \) a.e. on \( X \), thus in view of the Fatou Lemma

\[ \int_X |\Phi \circ f|d\mu = \int_X (\lim_{k \to \infty} |\Phi \circ f_{n_k}|)d\mu \leq \liminf_{k \to \infty} \int_X |\Phi \circ f_{n_k}|d\mu \leq \alpha, \]

so \( f \in A_\alpha(\Phi) \). \( \square \)

**Corollary 1.** The set \( A(\Phi) \) is an \( F_\sigma \)-subset of \((\mathcal{M}, \varrho)\).

**Proof.** It follows from Theorem 1, since \( A(\Phi) = \bigcup_{n=1}^{\infty} A_n(\Phi) \). \( \square \)

**Remark 1.** In the sequel we will use the fact that \( A(\Phi) \) is meager in \((\mathcal{M}, \varrho)\) if and only if \( \mathcal{M} \setminus A_\alpha(\Phi) \) is dense in \( \mathcal{M} \) for all \( \alpha > 0 \). Indeed, the sufficiency follows from Theorem 1 (resp. Corollary 1). Conversely, \((\mathcal{M}, \varrho)\) is a complete pseudo-metric space and therefore a Baire space as well (cf.[3],p.19), i.e. every nonempty open subset of \( \mathcal{M} \) is nonmeager in \((\mathcal{M}, \varrho)\). \( \square \)

Now we are prepared to determine the category of \( A(\Phi) \) in \( \mathcal{M} \).

**Theorem 2.** Suppose that

1. for each \( \varepsilon > 0 \) there exists \( E \in S \) such that \( 0 < \mu(E) < \varepsilon \).

Let \( \Phi \) be unbounded. Then \( A(\Phi) \) is meager in \((\mathcal{M}, \varrho)\).

**Proof.** Let \( f \in A_\alpha(\Phi) \) (where \( \alpha > 0 \)), \( \varepsilon > 0 \), further \( 0 < \mu(E) < \varepsilon \) for some \( E \in S \). Choose \( t_0 \in \mathbb{R} \) such that

\[ |\Phi(t_0)| > \frac{1}{\mu(E)} (\alpha - \int_{X \setminus E} |\Phi \circ f|d\mu). \]

Then for \( g = f \cdot \chi_{X \setminus E} + t_0 \cdot \chi_E \in \mathcal{M} \) we have

\[ \int_X |\Phi \circ g|d\mu = \int_{X \setminus E} |\Phi \circ f|d\mu + |\Phi(t_0)|\mu(E) > \alpha, \text{ thus } g \in \mathcal{M} \setminus A_\alpha(\Phi). \]

On the other hand \( E(f, g; \varepsilon) \subset E \), so \( g(f, g) < \varepsilon \) (see Remark 1). \( \square \)
Theorem 3. Let \((X, S, \mu)\) be a non-\(\sigma\)-finite measure space. Suppose that either \(\Phi\) is bounded or (1) does not hold.

Then \(A(\Phi)\) is meager in \((\mathcal{M}, g)\) if and only if \(|\Phi|^{-1}(0, +\infty) = \{t \in \mathbb{R}; |\Phi(t)| > 0\}\) is dense in \(\mathbb{R}\).

Proof. Suppose that \(|\Phi|^{-1}(0, +\infty)\) is dense in \(\mathbb{R}\). Let \(\alpha > 0\) and \(f \in A_\alpha(\Phi)\). Then \(f\) can be considered as a uniform limit of a sequence of elementary measurable functions \([2]p.86\). Hence we can find an elementary measurable function \(g = \sum_{n=1}^{\infty} a_n \chi_{E_n}\) (with \(X = \bigcup_{n=1}^{\infty} E_n\)) in every \(\varepsilon\)-neighbourhood of \(f\) in \((\mathcal{M}, g)\) \((\varepsilon > 0)\) such that \(\Phi(a_n) \neq 0\) for all \(n \in \mathbb{N}\).

Since \((X, S, \mu)\) is not \(\sigma\)-finite we can find \(m \in \mathbb{N}\) for which \(\mu(E_m) = +\infty\). It follows that

\[
\int_X |\Phi \circ g| d\mu \geq \int_{E_m} |\Phi \circ g| d\mu = |\Phi(a_m)| \mu(E_m) = +\infty,
\]
hence \(g \in \mathcal{M} \setminus A_\alpha(\Phi)\). Further see Remark 1.

Conversely, suppose that there exist \(\delta > 0, t \in \mathbb{R}\) such that \(\Phi(t') \equiv 0\), for every \(t' \in I = (t - \delta, t + \delta)\). Define \(f(x) \equiv t\), which is evidently in \(A(\Phi)\). Choose an arbitrary \(g \in \mathcal{M}\) from the \(\delta\)-neighbourhood of \(f\). Then we can find \(0 < r_0 < \delta\) such that \(E = E(f, g; r_0)\) is of measure less than \(\delta\). Then \(t - r_0 \leq g(x) \leq t + r_0\), consequently \(g(x) \in I\), thus

\[
(2) \quad \int_X |\Phi \circ g| d\mu = \int_{X \setminus E} |\Phi \circ g| d\mu + \int_E |\Phi \circ g| d\mu = \int_E |\Phi \circ g| d\mu = a.
\]

If (1) does not hold then \(a = 0\) for a suitably small \(\delta\), further if \(\Phi\) is bounded then \(a \leq K \mu(E) \leq K r_0 < +\infty\) for some \(K > 0\). It is now clear from (2) that under our assumptions \(\int_X |\Phi \circ g| d\mu < +\infty\), so \(g \in A(\Phi)\). Accordingly \(A(\Phi)\) contains a nonempty open ball. \(\square\)

Before we state the appropriate theorem for \(\sigma\)-finite spaces define the function

\[
\phi(c, \varepsilon) = \max_{t \in [c - \varepsilon, c + \varepsilon]} |\Phi(t)|, \quad \text{where } c \in \mathbb{R}, \varepsilon > 0.
\]

Theorem 4. Let \((X, S, \mu)\) be a \(\sigma\)-finite measure space and \(\{X_n\}_{n=1}^{\infty}\) be a measurable decomposition of \(X\) with \(\mu(X_n) < +\infty\). Suppose that either \(\Phi\) is bounded or (1) does not hold. Then \(A(\Phi)\) is meager in \((\mathcal{M}, g)\) if and only if

\[
(3) \quad \forall \varepsilon > 0 \ \forall c_n \in \mathbb{R} \ (n \in \mathbb{N}): \sum_{n=1}^{\infty} \mu(X_n) \cdot \phi(c_n, \varepsilon) = +\infty.
\]

Proof. First suppose that (3) holds. Choose arbitrary \(\alpha > 0, \varepsilon > 0\) and \(f \in A_\alpha(\Phi)\).

Examine \(f\) on the finite measure space \((X_n, S|_{X_n}, \mu|_{X_n})\) \((n \in \mathbb{N})\). There exists a sequence of simple measurable functions which converges a.e. to \(f\) on \(X_n\), further the convergence a.e. implies convergence in measure on finite measure spaces \([1]p.78\). It means that for every \(n \in \mathbb{N}\) there exists a simple measurable function \(g_n = \sum_{i=1}^{k(n)} c_{n,i} \chi_{X_n,i}\) (where \(k(n) \in \mathbb{N}, c_{n,i} \in \mathbb{R}, X_n,i \in S|_{X_n}\)) such that

\[
\mu(X_n \cap E(f, g_n; \frac{\varepsilon}{2})) \leq \frac{\varepsilon}{2^{n+1}}.
\]
Define the function \( g = \sum_{n=1}^{\infty} g_n \in \mathcal{M} \). We have

\[
\mu(E(f, g; \frac{\varepsilon}{2})) = \sum_{n=1}^{\infty} \mu(X_n \cap E(f, g; \frac{\varepsilon}{2})) \leq \\
\leq \sum_{n=1}^{\infty} \frac{\varepsilon}{2^{n+1}} = \frac{\varepsilon}{2}, \text{ so } g(f, g) \leq \frac{\varepsilon}{2}.
\]

For every \( n \in \mathbb{N} \) let \( c_n \) be that of the numbers \( c_{n,1}, \ldots, c_{n,k(n)} \) for which \( \phi(c_{n,i}, \frac{\varepsilon}{2}) \) is the least \( (1 \leq i \leq k(n)) \). Choose \( d_{n,i} \in [c_{n,i} - \frac{\varepsilon}{2}, c_{n,i} + \frac{\varepsilon}{2}] \) such that \( |\Phi(d_{n,i})| = \phi(c_{n,i}, \frac{\varepsilon}{2}) \) and put \( h = \sum_{n=1}^{\infty} \sum_{i=1}^{k(n)} d_{n,i} \chi_{X_n,i} \in \mathcal{M} \). Then \( \varrho(h, g) \leq \frac{\varepsilon}{2} \), thereby \( g(f, h) \leq g(f, g) + g(g, h) \leq \varepsilon \).

On the other hand from (3) we have

\[
\int_X |\Phi \circ h| d\mu = \sum_{n=1}^{\infty} \int_{X_n} |\Phi \circ h| d\mu = \sum_{n=1}^{\infty} \sum_{i=1}^{k(n)} \phi(c_{n,i}, \frac{\varepsilon}{2}) \cdot \mu(X_n,i) \geq \\
\geq \sum_{n=1}^{\infty} \sum_{i=1}^{k(n)} \phi(c_{n,i}, \frac{\varepsilon}{2}) \cdot \mu(X_n,i) = \sum_{n=1}^{\infty} \phi(c_n, \frac{\varepsilon}{2}) \cdot \sum_{i=1}^{k(n)} \mu(X_n,i) = \\
= \sum_{n=1}^{\infty} \phi(c_n, \frac{\varepsilon}{2}) \cdot \mu(X_n) = +\infty.
\]

It means that \( h \in \mathcal{M} \setminus A_\alpha(\Phi) \) (see Remark 1).

Conversely, if contrary to (3) \( \sum_{n=1}^{\infty} \phi(c_n, \varepsilon_0) \leq \alpha \) for some \( \alpha, \varepsilon_0 > 0 \) and \( c_n \in \mathbb{R} \) \( (n \in \mathbb{N}) \), then \( f = \sum_{n=1}^{\infty} c_n \chi_{X_n} \in A_\alpha(\Phi) \). Choose \( g \in \mathcal{M} \) such that \( g(f, g) < \delta \) \( (0 < \delta < \varepsilon_0) \). One can find an \( 0 < r_0 < \delta \), for which the measure of \( E = E(f, g; r_0) \) is less than \( \delta \).

We have

\[
\int_X |\Phi \circ g| d\mu = (\sum_{n=1}^{\infty} \int_{X_n \setminus E} |\Phi \circ g| d\mu) + \int_E |\Phi \circ g| d\mu \leq \\
\leq (\sum_{n=1}^{\infty} \int_{X_n \setminus E} \phi(c_n, \delta) d\mu) + \int_E |\Phi \circ g| d\mu \leq (\sum_{n=1}^{\infty} \phi(c_n, \varepsilon_0) \cdot \mu(X_n)) + \\
+ \int_E |\Phi \circ g| d\mu \leq \alpha + \int_E |\Phi \circ g| d\mu.
\]

Reasoning analogous to that of at the end of the proof of Theorem 3 works. \( \square \)

**Remark 2.** Observe that Theorems 2-4 determine the category of \( A(\Phi) \) in \((\mathcal{M}, \varrho)\) for every continuous \( \Phi \) and measure space \((X, S, \mu)\), respectively. However some of these theorems overlap, e.g. in one direction Theorem 3 holds for \( \sigma \)-finite measure spaces as well (the necessity of the density of \( |\Phi|^{-1}(0, +\infty) \) for \( A(\Phi) \) being meager), but in reverse it is false.

Indeed, let \((X, S, \mu)\) be an arbitrary \( \sigma \)-finite measure space. Let \( \{X_n\}_{n=1}^{\infty} \) be a measurable decomposition of \( X \) such that \( \mu(X_n) < +\infty \) for all \( n \in \mathbb{N} \). Define the sequence \( r_0 = 1, r_n = \frac{1}{2} \min\{r_{n-1}, \frac{1}{\mu(X_n)}\} \) if \( \mu(X_n) > 0 \) and \( r_n = \frac{1}{2} r_{n-1} \) if \( \mu(X_n) = 0 \) \((n \in \mathbb{N})\). Let
Then $\Phi$ is a nonincreasing, positive, bounded continuous function.

On the other hand setting $c_n = \frac{2n+1}{2} (n \in \mathbb{N})$ we get $\phi(c_n, \frac{1}{2}) = r_n$, thus $\sum_{n=1}^{\infty} \phi(c_n, \frac{1}{2}) \cdot \mu(X_n) \leq \sum_{n=1}^{\infty} \frac{1}{2^n} = 2$. Consequently by Theorem 4 $A(\Phi)$ is nonmeager in $(\mathcal{M}, g)$. □

**Corollary 2.** Let $p \geq 1$. Then $L_p(\mu)$ is nonmeager in $(\mathcal{M}, g)$ if and only if $\mu$ is finite and bounded away from zero.

**Proof.** Suppose that $\mu$ is not bounded away from zero (i.e. (1) holds). Since the function $\Phi(t) = |t|^p (p > 0)$ is continuous and unbounded Theorem 2 yields the desired result at once.

Assume now the converse of (1) and consider a non-$\sigma$-finite measure space $(X, S, \mu)$. Then $L_p(\mu)$ is meager in $(\mathcal{M}, g)$ by Theorem 3.

Suppose further that $(X, S, \mu)$ is $\sigma$-finite. Let $\{X_n\}_{n=1}^{\infty}$ be a measurable decomposition of $X$ with $\mu(X_n) < +\infty (n \in \mathbb{N})$. It is easy to check that $\phi(c, \varepsilon) \geq \varepsilon^p$ for all $\varepsilon > 0$ and $c \in \mathbb{R}$.

Consequently we get for every $c_n \in \mathbb{R} \ (n \in \mathbb{N})$ that

$$\sum_{n=1}^{\infty} \mu(X_n) \cdot \phi(c_n, \varepsilon) \geq \sum_{n=1}^{\infty} \mu(X_n) \cdot \varepsilon^p = \varepsilon^p \cdot \mu(X) = +\infty,$$

provided $\mu(X) = +\infty$. Then in virtue of Theorem 4 $L_p(\mu)$ is meager in $\mathcal{M}$.

Finally if $(X, S, \mu)$ is a finite measure space then putting $c_n = 0$ for all $n \in \mathbb{N}$ and $\varepsilon = 1$ we can see that (3) is not fulfilled, thus Theorem 4 completes the proof. □

**References**