SUPERPOROSITY IN A CLASS OF NON-NORMABLE SPACES

INTRODUCTION

The concept of porous set was introduced by Dolženko in [D]. Since then it has been thoroughly investigated and diversely generalized (see [Za1] or [Re] for a survey). It is possible to define several notions concerning porosity also in metric spaces (see [Za1], [Re]). It is known that in Banach spaces the ideal of meager sets is strictly wider than that of the \( \sigma \)-porous sets ([Za1]). It is true also in closed non-locally compact convex subsets of a separable Banach space ([AB]). Recently it has been established in dense in itself completely metrizable spaces as well (cf. [Za3]).

The primary goal of the research presented in this paper is in the line of the above results, i.e. to compare \( \sigma \)-porous and meager sets, respectively in some non-normable spaces. Such an attempt was made in [TZs] where the space \( s \) of all real sequences endowed with the Fréchet metric

\[
\rho_F(\{a_n\}_n, \{b_n\}_n) = \sum_n 2^{-n} \frac{|a_n - b_n|}{1 + |a_n - b_n|}
\]

where \( \{a_n\}_n, \{b_n\}_n \in s \) was scrutinized in this respect. This space is non-normable ([KG], Exercise 276) and it was shown in [TZs] e.g. that the set \( \{\{a_n\}_n \in s; \sum_n \Phi(a_n) \text{ converges} \} \) is \( \sigma \)-superporous in \( s \) for a residual family of functions \( \Phi \) in the space of all real functions furnished with the uniform topology.

It is the purpose of this paper to carry on these investigations generalizing results of [TZs] for the space \( M \) of all measurable functions on an infinite \( \sigma \)-finite measure space \( (X, S, \mu) \) endowed with the (metrizable) topology of convergence in measure on sets of finite measure (see [G]). We will show that results quite analogous to those of exposed in [TZs] for \( s \) hold in this generality as well. For instance, the set \( A(\Phi) = \{f \in M; \int_X |\Phi \circ f|d\mu^* < +\infty \} \) is \( \sigma \)-superporous in \( M \) for a broad class of functions \( \Phi : \mathbb{R} \to \mathbb{R} \), where \( \mu^* \) is the outer measure induced by \( \mu \) and \( \int_X h d\mu^* \) stands for the \( \mu^* \)-upper integral of the function \( h : X \to \mathbb{R} \) (see [F], Section 2.4).

Further we show that \( A(\chi_{\mathbb{R} \setminus M}) \) is \( \sigma \)-superporous in \( M \) for every \( \sigma \)-very porous set \( M \subset \mathbb{R} \) (\( \chi_{\mathbb{R} \setminus M} \) is the characteristic function of \( \mathbb{R} \setminus M \)) and that \( A(\chi_{\mathbb{R} \setminus M}) \) is meager in \( M \) if \( M \) is meager at some point of \( \mathbb{R} \). In particular, \( A(\chi_{\mathbb{R} \setminus M}) \) is meager in \( (s, \rho_F) \) if and only if \( M \) is meager at some point of \( \mathbb{R} \).

This could provide a method for relating meager non-\( \sigma \)-porous subsets of \( \mathbb{R} \) to meager non-\( \sigma \)-porous subsets of \( M \) (resp. \( s \)) if the porosity of \( A(\chi_{\mathbb{R} \setminus M}) \) in \( M \) (resp. \( s \)) could be characterized in terms of \( M \subset \mathbb{R} \).

It is worth noticing here that a more familiar metrization of \( M \) by the metric

\[
m(f, g) = \inf \{ \varepsilon > 0; \mu(\{x \in X; |f(x) - g(x)| \geq \varepsilon \}) < \varepsilon \} \quad (f, g \in M)
\]
which coincides with the topology of convergence in measure on $X$ (cf. [F], Section 2.3.8), yields a setting where our considerations are not feasible even for continuous $\Phi$’s. This question was studied in [Zs1].

Preliminaries

In the sequel $(X, S, \mu)$ will be an infinite $\sigma$-finite measure space and $\mu^*$ the outer measure induced by $\mu$. Without loss of generality we may suppose that $X = \bigcup_{n=1}^{\infty} X_n$, where $\{X_n\}_{n=1}^{\infty}$ is a sequence of pairwise disjoint, $S$-measurable sets such that $2 < \mu(X_n) < +\infty$ for each $n \in \mathbb{N}$.

Denote by $\mathcal{M}$ (resp. $\mathcal{M}_n$) the set of all $S$-measurable functions that are finite almost everywhere (abbr. a.e.) on $X$ (on $X_n$). We will identify members of $\mathcal{M}$ provided they equal a.e. on $X$.

If the sequence $f_k \in \mathcal{M}$ ($k \in \mathbb{N}$) converges in measure to $f \in \mathcal{M}$, write $f_k \overset{\mu}{\longrightarrow} f$.

Denote by $\mathcal{F}_m$ the space of all functions $\Phi : \mathbb{R} \to \mathbb{R}$ such that $\Phi \circ f \in \mathcal{M}$ for all $f \in \mathcal{M}$. It is known that $\mathcal{F}_m$ contains the class of Borel-measurable functions. Observe that $\mathcal{F}_m$ is a closed subspace of the complete metric space $(\mathcal{F}, d)$, where $\mathcal{F} = \mathbb{R}^\mathbb{R}$ and

$$d(\Phi, \Psi) = \min\{1, \sup_{t \in \mathbb{R}}|\Phi(t) - \Psi(t)|\} \quad (\Phi, \Psi \in \mathcal{F}).$$

Indeed, if a sequence $\Phi_n \in \mathcal{F}_m$ ($n \in \mathbb{N}$) $d$-converges to $\Phi \in \mathcal{F}$ then $\Phi_n \circ f \in \mathcal{M}$ for all $f \in \mathcal{M}$, thus $\Phi \circ f \in \mathcal{M}$ and consequently $\Phi \in \mathcal{F}_m$. It follows that $(\mathcal{F}_m, d)$ is a complete metric space.

For $\Phi \in \mathcal{F}$ and $p \in \mathbb{N}$ define

$$A(\Phi) = \{f \in \mathcal{M}; \int_X |\Phi \circ f|d\mu^* < +\infty\} \quad \text{and}$$

$$A_p(\Phi) = \{f \in \mathcal{M}; \int_X |\Phi \circ f|d\mu^* \leq p\},$$

where $\int_X f d\mu^*$ is the upper integral of $f$ with respect to $\mu^*$ (see [F], Section 2.4).

For $f, g \in \mathcal{M}$ and $n \in \mathbb{N}$ define

$$\rho_n(f, g) = \int_{X_n} \frac{|f - g|}{1 + |f - g|} d\mu$$

$$\rho(f, g) = \sum_{n=1}^{\infty} \frac{1}{2^n \mu(X_n)} \rho_n(f, g).$$

For $i, j \in \mathbb{N}$ and $M \subset \mathbb{R}$ denote

$$\tilde{A}_{i,j}(M) = \{f \in \mathcal{M}; \mu^*(f^{-1}(M) \cap X_i) \geq \frac{\mu(X_i)}{j}\},$$

$$A_{i,j}(M) = \{f|_{X_i}; f \in \tilde{A}_{i,j}(M)\} \quad \text{and} \quad A_{i,0}(M) = \{f \in \mathcal{M}; \mu^*(f^{-1}(M)) = \mu(X_i)\}.$$ 

It is not hard to see that $\rho$ (resp. $\rho_n$) is a metric on $\mathcal{M}$ (resp. $\mathcal{M}_n$). It can be shown similarly as for $(s, p)$ that $(\mathcal{M}, \rho)$ is non-normable (see [Zs2]).

Convergence in measure implies $\rho$-convergence and the converse holds if and only if the underlying measure space is finite. More precisely we have:
Lemma 1. Let $f_k, f \in \mathcal{M} \ (k \in \mathbb{N})$. The following are equivalent:

(i) $f_k \overset{\rho}{\rightarrow} f$;
(ii) $f_k \overset{\mu}{\rightarrow} f$ on every $S$-measurable set of finite measure;
(iii) $f_k|_{X_n} \overset{\rho_n}{\rightarrow} f|_{X_n}$ for all $n \in \mathbb{N}$.

Proof. For (i)$\Leftrightarrow$(ii) see [G], Theorem 3. The equivalence (i)$\Leftrightarrow$(iii) follows easily from [K] (Theorem 14, p.122).

Remark 1. Observe that $(\mathcal{M}_n, \rho_n)$ is a complete metric space for each $n \in \mathbb{N}$ and the $\rho_n$-convergence of sequences from $\mathcal{M}_n$ coincides with their convergence in measure on $X_n$ ([Ha], Problem 42(4)). Further the equivalence (i)$\Leftrightarrow$(iii) in the previous lemma actually yields that $(\mathcal{M}, \rho)$ and the Tychonoff product $\Pi_n(\mathcal{M}_n, \rho_n)$ are homeomorphic.

Lemma 2. (cf.[G]) $(\mathcal{M}, \rho)$ is a complete metric space.

Denote by $B_d(y, r)$ the open ball about $y \in Y$ with radius $r > 0$ in the metric space $(Y, d)$. By $B(x, r)$ we will denote the interval $(x - r, x + r)$, where $x \in \mathbb{R}$. For $E \subset Y, y \in Y$ and $r > 0$ define

$$\gamma(y, r, E) = \sup\{r' > 0; \exists y' \in Y B_d(y', r') \subset B_d(y, r) \setminus E\}.$$ 

We say that $E$ is porous (very porous) at $y$ if

$$\limsup_{r \to 0+} \frac{\gamma(y, r, E)}{r} > 0 \ (\liminf_{r \to 0+} \frac{\gamma(y, r, E)}{r} > 0).$$

Further $E$ is said to be superporous at $y \in Y$ (see [Za1],[Za2]), if $E \cup F$ is porous at $y$ whenever $F \subset Y$ is porous at $y$.

A set $E \subset Y$ is said to be globally very porous if there exist constants $0 < a_E < 1$ and $r_0 > 0$ such that $\gamma(y, r, E) > a_E r$ for every $y \in E$ and $0 < r < r_0$ ([Za1]).

We say that $E$ is superporous (very porous) if it is superporous (very porous) at each of its points, further $E$ is $\sigma$-superporous ($\sigma$-very porous) if it is a countable union of superporous (very porous) sets. Superporosity was defined in [Za2] in connection with the $\mathcal{I}$-density topology of Wilczynski and others (cf.[W]).

Note that superporosity implies very porosity as observed in [Za2] (see [Re], Corollary 8.15 as well) and $\sigma$-superporosity is equivalent to $\sigma$-very porosity which is further equivalent to $\sigma$-globally very porosity ([Re], Corollary 8.17).

We will denote by $\text{card} Y$ and $\mathcal{P}(Y)$ the cardinality and the power set, respectively of the set $Y$, further $c$ will stand for the power of the continuum. Denote by $|I|$ the length of the interval $I \subset \mathbb{R}$.

Main Results

Lemma 3. Let $\{I_q; q \in \mathbb{N}\}$ be an enumeration of open intervals with rational endpoints. Let $\Phi_{pq} = p\chi_{I_q}$ for $p, q \in \mathbb{N}$. Then $A_p(\Phi_{pq})$ is superporous in $(\mathcal{M}, \rho)$ for every $p, q \in \mathbb{N}$.

Proof. Choose $p, q \in \mathbb{N}$ and denote by $t_q$ the midpoint of $I_q$. Let $f \in A_p(\Phi_{pq})$. Suppose that $F \subset \mathcal{M}$ is an arbitrary set porous at $f$. Then there exist sequences
\[ r_n, r'_n > 0 \ (n \in \mathbb{N}) \] and \( \alpha > 0 \) such that \( \alpha r_n < r'_n < r_n < 2^{-n}, \) further we get an \( f_n \in \mathcal{M} \) such that

\[ B_\rho(f_n, r'_n) \subset B_\rho(f, r_n) \setminus F. \]

Define \( p_n = \min\{k \in \mathbb{N}; 2^{-k} < r'_n\} + 1 \) and \( \varepsilon_n = 2^{-p_n+1} \) for all \( n \in \mathbb{N}. \) Then we have

\[ r'_n > \varepsilon_n \geq \frac{r'_n}{2}. \]

Denote \( E_{n1} = X_{\rho n} \cap f_0^{-1}((t_q - \frac{1}{2}|I_q|, t_q + \frac{1}{2}|I_q|)) \) and \( E_{n2} = X_{\rho n} \setminus E_{n1} \) and define \( g_n = f_n \chi_{X \setminus X_{\rho n}} + t_q \chi_{E_{n2}} + (t_q + \frac{1}{2}|I_q|) \chi_{E_{n1}} \in \mathcal{M}. \) It is clear that

\[ |f_n(x) - g_n(x)| \geq \frac{1}{8}|I_q| \quad \text{for all} \quad x \in X_{\rho n}. \]

Since \( \rho(f_n, g_n) = \frac{1}{2^{p_n} \mu(X_{\rho n})} \int_{X_{\rho n}} \frac{|f_n - g_n|}{1 + |f_n - g_n|} \, d\mu \) then by the definition of \( \varepsilon_n, X_{\rho n} \) and (3), respectively we get

\[
\rho(f_n, g_n) < \varepsilon_n \quad \text{and} \quad \rho(f_n, g_n) > \frac{|I_q|}{8 + |I_q|} \varepsilon_n = \frac{2 \delta_n \mu(D_{n0})}{\mu(X_{\rho n})},
\]

Put \( \delta_n = \frac{|I_q|}{8 + |I_q|} \rho(f_n, g_n) \) and pick an arbitrary \( h_n \in B_\rho(g_n, \delta_n). \) Define

\[ D_n = \{x \in X_{\rho n}; |h_n(x) - g_n(x)| < \frac{4 \delta_n}{\varepsilon_n - 4 \delta_n}\} \quad \text{and} \quad D_{n0} = X_{\rho n} \setminus D_n. \]

Observe that \( D_n \) is well-defined, since by (4) \( \delta_n = \frac{|I_q|}{8 + |I_q|} \rho(f_n, g_n) \) is always positive.

Then we have

\[
\delta_n > \rho(h_n, g_n) \geq \frac{1}{2^{p_n} \mu(X_{\rho n})} \int_{D_{n0}} \frac{|h_n - g_n|}{1 + |h_n - g_n|} \, d\mu \geq \frac{\varepsilon_n}{2^{p_n} \mu(X_{\rho n})} \int_{D_{n0}} \frac{4 \delta_n}{\varepsilon_n} \, d\mu = \frac{2 \delta_n \mu(D_{n0})}{\mu(X_{\rho n})},
\]

thus \( \mu(D_{n0}) < \frac{1}{2^{p_n}} \mu(X_{\rho n}), \) hence \( \mu(D_{n0}) \geq \frac{1}{2} \mu(X_{\rho n}) > 1. \)

In view of (4) we get \( |h_n(x) - g_n(x)| < \frac{4 \delta_n}{\varepsilon_n - 4 \delta_n} < \frac{1}{8}|I_q| \) for every \( x \in D_n, \) so \( h_n(D_n) \subset (t_q - \frac{3}{2}|I_q|, t_q + \frac{3}{2}|I_q|) \) (see the definition of \( g_n). \) Then \( \int_X |\Phi_q \circ h_n| \, d\mu^* \geq \int_{D_n} |\Phi_q \circ h_n| \, d\mu^* \geq \rho(D_{n0}) > \rho, \) so

\[ h_n \in \mathcal{M} \setminus A_p(\Phi_q). \]

Using (4) we get \( |\varepsilon_n - \rho(f_n, g_n)| > \frac{\varepsilon_n}{2} > \frac{\varepsilon_n}{8 + |I_q|} > \delta_n, \) therefore \( B_\rho(g_n, \delta_n) \subset B_\rho(f_n, \varepsilon_n) \subset B_\rho(f_n, r'_n). \) Then in virtue of (5) and (1) there holds

\[ B_\rho(g_n, \delta_n) \subset B_\rho(f_n, r'_n) \setminus A_p(\Phi_q) \subset B_\rho(f, r_n) \setminus (F \cup A_p(\Phi_q)). \]

From (4') and (2) we get

\[
\gamma(f, r_n, F \cup A_p(\Phi_q)) \geq \delta_n \geq \left( \frac{|I_q|}{8 + |I_q|} \right)^2 \frac{\varepsilon_n}{2} \geq \left( \frac{|I_q|}{8 + |I_q|} \right)^2 \frac{r'_n}{4} \geq \left( \frac{|I_q|}{8 + |I_q|} \right)^2 \frac{\alpha}{4} r_n,
\]

thus \( \limsup_{r \to 0^+} \gamma(f, r, F \cup A_p(\Phi_q)) \geq \left( \frac{|I_q|}{8 + |I_q|} \right)^2 \frac{\alpha}{4} > 0, \) which proves the porosity of \( F \cup A_p(\Phi_q) \) at \( f. \) \( \square \)
Theorem 1. Let $\Phi \in \mathcal{F}$ be a function for which there exists $t_0 \in \mathbb{R} \cup \{\pm \infty\}$ such that
\begin{equation}
\liminf_{t \to t_0} |\Phi(t)| > 0.
\end{equation}
Then $A(\Phi)$ is $\sigma$-superporous in $(\mathcal{M}, \rho)$.

Proof. In view of (6) there exists $\beta > 0$ and a bounded open interval $I$ such that
\begin{equation}
|\Phi(t)| \geq \beta \quad \text{for all } t \in I.
\end{equation}
Let $\{J_k; k \in \mathbb{N}\}$ be a partition of $I$ consisting of open intervals. Choose an $f \in A(\Phi)$. Then by (7) we have
\begin{equation}
\beta \sum_{k \in \mathbb{N}} \mu(f^{-1}(J_k)) = \beta \mu(f^{-1}(I)) \leq \int_X |\Phi \circ f| d\mu^* < p
\end{equation}
for some $p \in \mathbb{N}$. Thus $\mu(f^{-1}(J_k)) \leq 1$ for some $k \in \mathbb{N}$ and hence $\mu(f^{-1}(I_q)) \leq 1$ for some open interval $I_q \subset J_k$ with rational endpoints. Consequently,
\begin{equation}
\int_X |\Phi_{pq} \circ f| d\mu^* = p\mu(f^{-1}(I_q)) \leq p,
\end{equation}
so $f \in A_{pq}(\Phi_{pq})$, whence $A(\Phi) \subset \bigcup_{p,q \in \mathbb{N}} A_{pq}(\Phi_{pq})$, which concludes the proof by Lemma 3.

As the following results show, there are also functions $\Phi$, not necessarily satisfying (6), for which $A(\Phi)$ is still $\sigma$-superporous (cf. Theorem 2):

Lemma 4. Let $\mathcal{M} \subset \mathbb{R}$ be a globally very porous set. Then $\tilde{A}_{i,j}(\mathcal{M})$ is superporous in $(\mathcal{M}, \rho)$ for each $i, j \in \mathbb{N}$.

Proof. According to the assumption on $\mathcal{M}$ there exist $0 < a_M < 1$ and $r_0 > 0$ such that
\begin{equation}
\gamma(x, r, \mathcal{M}) > a_M r \quad \text{for all } x \in \mathcal{M} \cup (\mathbb{R} \setminus \overline{\mathcal{M}}) \text{ and all } 0 < r < r_0.
\end{equation}
Choose $f \in \tilde{A}_{i,j}(\mathcal{M})$ and a set $F \subset \mathcal{M}$ which is porous at $f$. Then there exist $\alpha > 0$, sequences $r_n, r'_n > 0$ and $f_n \in \mathcal{M}$ such that $r_n \prec 0, \alpha r_n < r'_n < r_n < 2^{-i+1}, \frac{3r_0}{1+r_0}$ and
\begin{equation}
B(f_n, r'_n) \subset B(f, r_n) \setminus F.
\end{equation}
It is not hard to find $b_{nk} \in \mathbb{R}$ ($1 \leq k \leq m_n$, where $m_n \in \mathbb{N}$) and a partition $\{D_{nk}; 1 \leq k \leq m_n\}$ of $X_i$ such that for $g_{n0} = f_n \chi_{X \setminus X_i} + \sum_{k=1}^{m_n} b_{nk} \chi_{D_{nk}} \in \mathcal{M}$ there holds
\begin{equation}
\rho(f_n, g_{n0}) < \frac{r'_n}{4}.
\end{equation}
We can actually choose \( b_{nk} \in M \cup (\mathbb{R} \setminus \overline{M}) \) for every \( 1 \leq k \leq m_n \).

Put \( \eta_n = \frac{2^{1 + \frac{r_n}{2}}}{6 - 2r_n} \). Then \( \eta_n < r_0 \), so it follows from (8) that for each \( 1 \leq k \leq m_n \) there exists \( b_{nk}' \in \mathbb{R} \) and \( r_{nk} > 0 \) such that

\[
(11) \quad a_M \eta_n \leq r_{nk} < \eta_n \quad \text{and} \quad B(b_{nk}', r_{nk}) \subset B(b_{nk}, \eta_n) \setminus M.
\]

Define \( g_n = g_{a_0} \chi \setminus X, + \sum_{k=1}^{m_n} b_{nk}' \chi D_{nk} \in M \). Then by (11) we have

\[
\rho(g_{a_0}, g_n) \leq \frac{1}{2\mu(X)} \sum_{k=1}^{m_n} \int_{D_{nk}} |b_{nk} - b_{nk}'|d\mu = \frac{1}{2\mu(X)} \sum_{k=1}^{m_n} \|b_{nk} - b_{nk}'\|_\mu (D_{nk}) \leq \frac{1}{2r_n} \sum_{k=1}^{m_n} \mu(D_{nk}) = \frac{r'_n}{6 - 2r_n} \leq \frac{r'_n}{4},
\]

thus in view of (10)

\[
(12) \quad \rho(f_n, g_n) \leq \rho(f_n, g_{a_0}) + \rho(g_{a_0}, g_n) \leq \frac{r'_n}{2}.
\]

We have \( 0 < a_M < 1 < 3j \), thus \( \frac{r'_n}{2} > \frac{a_M r'_n}{6j} \). Then putting \( \delta_n = \frac{a_M r'_n}{6j} \) we get by (12) that \( r'_n - \rho(f_n, g_n) \geq \frac{r'_n}{2} > \delta_n \), so

\[
(13) \quad B_\rho(g_n, \delta_n) \subset B_\rho(f_n, r'_n).
\]

Choose \( h \in \tilde{A}_{i,j}(M) \) arbitrarily. According to (11) we have

\[
\rho(h, g_n) \geq \frac{1}{2\mu(X)} \int_{h^{-1}(M) \cap X_i} \frac{|h - g_n|^2}{1 + |h - g_n|^2} d\mu \geq \frac{1}{2\mu(X)} \sum_{1 \leq k \leq m_n} \frac{r_{nk}}{1 + \min_{1 \leq k \leq m_n} \frac{r_{nk}}{r_{nk}}} \geq \frac{1}{2\mu(X)} \frac{\mu(X_i)}{j} \frac{a_M \eta_n}{1 + a_M \eta_n} > \frac{1}{2j} \cdot \frac{a_M \eta_n}{1 + \eta_n} = \delta_n.
\]

It means by (13) that \( B_\rho(g_n, \delta_n) \subset B_\rho(f_n, r'_n) \setminus \tilde{A}_{i,j}(M) \). Then in virtue of (9) we get \( B_\rho(g_n, \delta_n) \subset B_\rho(f, r'_n) \setminus (F \cup \tilde{A}_{i,j}(M)) \). Consequently

\[
\gamma(f, r'_n, F \cup \tilde{A}_{i,j}(M)) \geq \delta_n > \frac{a_M \alpha r_n}{6j},
\]

which justifies the porosity of \( F \cup \tilde{A}_{i,j}(M) \) at \( f \). \( \square \)

**Theorem 2.** Let \( M \) be a \( \sigma \)-very porous set. Then \( A(\chi_{\mathbb{R} \setminus M}) \) is \( \sigma \)-superporous in \((M, \rho)\).

**Proof.** We may already suppose that \( M = \bigcup_{k=1}^\infty M_k \), where \( M_k \) is globally very porous and \( a_M < 1 \) for all \( k \in \mathbb{N} \).

Choose \( f \in A(\chi_{\mathbb{R} \setminus M}) \). Then we have
generality we may assume that
by the metric
with the topology of convergence in measure on
In the sequel we will use that the topology induced by

Therefore

Finally, denote

which concludes the proof by Lemma 4. □

Now we turn to characterizing the meagerness of \( A(\chi_{\mathbb{R}\setminus M}) \) in \((\mathcal{M}, \rho)\) in terms of properties of \( M \). We will need the following

Lemma 5. If \( M \) is meager at some point of \( \mathbb{R} \), then \( A_{i,j}(M) \) is meager at some point of \((\mathcal{M}_i, \rho_i)\) for all \( i, j \in \mathbb{N} \).

Proof. In the sequel we will use that the topology induced by \( \rho_i \) on \( \mathcal{M}_i \) is equivalent with the topology of convergence in measure on \( X_i \), i.e. with the topology induced by the metric \( m_i = m|_{\mathcal{M}_i \times \mathcal{M}_i} \) (see [Ha], Problem 42(4)).

Suppose that there exists an interval \( U = B(t_0, r) \) \((t_0 \in \mathbb{R}, r > 0)\) such that \( U \cap M = \bigcup_{k=1}^{\infty} M_k \) for some nowhere dense sets \( M_k \subset \mathbb{R} \) \((k \in \mathbb{N})\). Without loss of generality we may assume that \( M_k \subset M_{k+1} \) for all \( k \in \mathbb{N} \). Let \( f_0 \equiv t_0 \) on \( X_i \) and put \( V = B_{m_i}(f_0, r) \).

We will show that \( V \cap A_{i,j}(M_k) \) is nowhere dense in \((\mathcal{M}_i, m_i)\): take an open ball \( B_{m_i}(f, \varepsilon) \) in \( \mathcal{M}_i \). We may already suppose that \( f \in V \) and \( f \) equals a simple function \( \sum_{s=1}^{m} b_s \chi_{D_s} \) where \( b_1, \ldots, b_m \in U \) and \( D_1, \ldots, D_m \) is a measurable partition of \( X_i \).

Then the nowhere density of \( M_k \) in \( \mathbb{R} \) yields some \( b'_s \in \mathbb{R} \) and \( 0 < \varepsilon_0 < \frac{\mu(X_i)}{j} \) such that

\[
B(b'_s, \varepsilon_0) \subset B(b_s, \varepsilon) \setminus M_k \quad \text{for any } 1 \leq s \leq m.
\]

Choose \( g \in B_{m_i}(f_1, \varepsilon_0) \) where \( f_1 = \sum_{s=1}^{m} b'_s \chi_{D_s} \); then by (14)

\[
g^{-1}(M_k) \subset \{ x \in X_i ; |f_1(x) - g(x)| \geq \varepsilon_0 \}.
\]

Therefore \( \mu^*(g^{-1}(M_k)) \leq \varepsilon_0 < \frac{\mu(X_i)}{j} \), so \( g \notin A_{i,j}(M_k) \). On the other hand \( f_1 \in B_{m_i}(f, \varepsilon) \); thus,

\[
0 \neq B_{m_i}(f, \varepsilon) \cap B_{m_i}(f_1, \varepsilon_0) \subset B_{m_i}(f, \varepsilon) \setminus A_{i,j}(M_k),
\]

which justifies the nowhere density of \( V \cap A_{i,j}(M_k) \) in \( \mathcal{M}_i \).

Finally, denote \( V_0 = B_{m_i}(f_0, r_0) \) where \( r_0 = \min\{r, \frac{1}{j} \} \). Pick \( h \in A_{i,j}(M) \cap V_0 \). Then \( h^{-1}(M \setminus U) \subset \{ x \in X_i ; |h(x) - f_0(x)| \geq r_0 \} \), so \( \mu^*(h^{-1}(M \setminus U)) \leq r_0 \leq \frac{1}{j} < \frac{\mu(X_i)}{2j} \). Furthermore in view of the regularity of \( \mu^* \) we get (cf. [F], Section 2.1.5(1))

\[
\frac{\mu(X_i)}{j} \leq \mu^*(h^{-1}(M)) \leq \mu^*(h^{-1}(M \cap U)) + \mu^*(h^{-1}(M \setminus U)) < \\
< \lim_{k \to \infty} \mu^*(h^{-1}(M_k)) + \frac{\mu(X_i)}{2j},
\]
hence \( \lim_{k \to \infty} \mu^*(h^{-1}(M_k)) > \frac{\mu(X)}{2^j} \), so \( h \in A_{i,2j}(M_k) \cap V_0 \subset A_{i,2j}(M_k) \cap V \) for some \( k \in \mathbb{N} \). Therefore
\[
A_{i,j}(M) \cap V_0 \subset \bigcup_{k=1}^{\infty} A_{i,2j}(M_k) \cap V
\]
which means that \( A_{i,j}(M) \) is meager at \( f_0 \) in \( M_i \).

**Theorem 3.** If \( M \) is meager at some point of \( \mathbb{R} \) then \( A(\chi_{\mathbb{R}\setminus M}) \) is meager in \( (M, \rho) \).

**Proof.** Let \( t_0 \in \mathbb{R} \) and \( r > 0 \) be such that \( B(t_0, r) \cap M \) is meager in \( \mathbb{R} \). Let \( V_i = B_m(f_0, r_0) \) where \( f_0 \equiv t_0 \) on \( X \) and \( 0 < r_0 = \min\{r, \frac{1}{i}\} \). Then by Lemma 5 \( A_{i,2j}(M) \cap V_i \) is meager in \( (M_i, \rho_i) \) for all \( i \in \mathbb{N} \).

Choose \( f \in A(\chi_{\mathbb{R}\setminus M}) \). Then \( \mu^*(f^{-1}(\mathbb{R} \setminus M)) < +\infty \) and by the regularity of \( \mu^* \) there exists a \( \mu^* \)-hull \( B \) of \( f^{-1}(\mathbb{R} \setminus M) \) (see [F], Section 2.1.4). Consequently, \( \mu(B \cap X_i) = \mu^*(f^{-1}(\mathbb{R} \setminus M) \cap X_i) = \mu^*(X_i \setminus (X_i \cap f^{-1}(M))) \); thus,
\[+\infty > \mu^*(f^{-1}(\mathbb{R} \setminus M)) = \mu(B) = \sum_{i=1}^{\infty} \mu(B \cap X_i) = \sum_{i=1}^{\infty} \mu^*(X_i \setminus (X_i \cap f^{-1}(M))).\]

Then for all \( i \geq m \) \((m \in \mathbb{N})\) we have
\[
\frac{\mu(X_i)}{2} > 1 > \mu^*(X_i \setminus (X_i \cap f^{-1}(M))) \geq \mu(X_i) - \mu^*(X_i \cap f^{-1}(M)),
\]
hence \( f|_{X_i} \in A_{i,2}(M) \) for all \( i \geq m \). Accordingly,
\[A(\chi_{\mathbb{R}\setminus M}) \subset \bigcup_{m=1}^{\infty} P_m \text{ where } P_m = \prod_{i=1}^{m-1} M_i \times \prod_{i=m}^{\infty} A_{i,2}(M) \text{ for each } m \in \mathbb{N}.\]

It suffices now to show by Remark 1 that \( P_m \) is meager in \( P = \prod_{i=1}^{\infty} M_i \) for every \( m \in \mathbb{N} \). Let \( U = \prod_{i=1}^{m-1} U_i \times \prod_{i=m}^{\infty} M_i \) be any basic open set of the product topology on \( P \) such that \( n \geq m \). Denote by \( V \) the open set \( \prod_{i=1}^{m-1} U_i \times V_{n+1} \times \prod_{i=n+2}^{\infty} M_i \subset P \). Then \( V \subset U \) and \( V \cap P_m \subset \prod_{i=1}^{m-1} U_i \times (V_{n+1} \cap A_{n+1,2}(M)) \times \prod_{i=n+2}^{\infty} A_{i,2}(M) \), which is meager in \( P \). It means by Theorem 1.7. in [HMC] that \( P_m \) is meager in \( P \). \( \square \)

**Corollary.** \( A(\chi_{\mathbb{R}\setminus M}) \) is meager in \((s, \rho_F)\) if and only if \( M \) is meager at some point of \( \mathbb{R} \).

**Proof.** The sufficiency follows from the previous theorem by putting \( X = \mathbb{N} \), \( S = \mathcal{P}(\mathbb{N}) \) and the counting measure on \( \mathbb{N} \) for \( \mu \).

Conversely, suppose that \( M \) is non-meager everywhere in \( \mathbb{R} \). Then \( M \) with the relative topology is a dense Baire subspace of \( \mathbb{R} \). Then the product \( E = M^\mathbb{N} \) is a Baire space which is clearly dense in \( s \) ([HMC], Lemma 5.6.). Therefore \( E \) is non-meager in \( s \) and hence \( A(\chi_{\mathbb{R}\setminus M}) \supset E \) is non-meager in \( s \). \( \square \)

**Remark 2.** In connection with the Corollary a question arises if a similar characterization of \( A(\chi_{\mathbb{R}\setminus M}) \) is possible also in \( M \). Mimicking the above proof and using Remark 1 it would be sufficient to prove that non-meagerness of \( M \) everywhere in \( \mathbb{R} \) implies non-meagerness of \( A_{i,0} \) everywhere in \( M_i \) for each \( i \in \mathbb{N} \), further that \( M_i \) is separable for each \( i \in \mathbb{N} \). This last condition is needed for the theorem on product of Baire spaces ([HMC], Lemma 5.6.), thus we may consider the question only for separable measure spaces \((X, S, \mu)\) (see [Ha], §41).
It is not hard to show that this is really the case if each $X_i$ is a finite disjoint sum of atoms, however in general the answer is not known to me.

**Remark 3.** Another question here arises in connection with finding necessary conditions for $\sigma$-porosity of $A(\chi_{\mathbb{R}\setminus M})$ in $M$ (or at least in $s$). If we want to use some argument similar to that of in the Corollary, we would need some “porosity-Baire” product theorem as the mentioned result of Oxtoby ([O1],[HMC]). This ultimately breaks down to proving a porosity version of the well-known Kuratowski-Ulam theorem on sections of nowhere dense subsets of the product space ([O2], Theorem 15.1). More precisely, the questions are as follows:

(i) If $X$ and $Y$ are separable metric spaces and $E$ is a porous subset of $X \times Y$ with (say) the box metric, then are the $x$-sections $E_x$ of $E$ porous in $Y$ except for a $\sigma$-porous set in $X$?

(ii) Call a metric space $Z$ $p$-Baire if every nonempty open subset of $Z$ is non-$\sigma$-porous. Is the property of being separably $p$-Baire (countably) productive?

The preceding theorems provide sufficient background for investigating the class

$$\mathcal{U} = \{ \Phi \in \mathcal{F}; A(\Phi) \text{ is } \sigma\text{-superporous in } (\mathcal{M}, \rho) \}.$$}

**Theorem 4.** We have

(i) $\text{card}(\mathcal{U} \cap \mathcal{F}_m) = \text{card}\mathcal{U} = 2^c$

(ii) $\text{card}(\mathcal{F} \setminus \mathcal{U}) = 2^c$ for $(s, \rho_F)$.

**Proof.**

(i) Every subset of the Cantor’s ternary set $C$ is very porous therefore in view of Theorem 2 $\chi_{\mathbb{R}\setminus E} \in \mathcal{U} \cap \mathcal{F}_m$ for every $E \subset C$, further $\chi_{\mathbb{R}\setminus E} \neq \chi_{\mathbb{R}\setminus E'}$ provided $E \neq E'$. Consequently $\text{card}(\mathcal{U} \cap \mathcal{F}_m) \geq \text{card}\mathcal{P}(C) = 2^c$. Further clearly $\text{card}\mathcal{U} \leq \text{card}\mathcal{F} \leq \text{card}\mathbb{R}^c = 2^c$.

(ii) If we restrict ourselves to $(s, \rho_F)$ only, then $\chi_E \notin \mathcal{U}$ for each $E \subset C$ since $A(\chi_E) = s \setminus A(\chi_{\mathbb{R}\setminus E})$ and $(s, \rho_F)$ is a nonmeager space by Lemma 2. Thus again $2^c = \text{card}\mathcal{P}(C) \leq \text{card}(\mathcal{F} \setminus \mathcal{U}) \leq \text{card}\mathcal{F} \leq 2^c$. $\square$

Further we have

**Theorem 5.** $\mathcal{U}$ is residual in $\mathcal{F}$.

**Proof.** See [TZs], Lemma 2 and our Theorem 1. $\square$

**Remark 4.** It is worth noticing that if we restrict our investigations onto $\mathcal{F}_m$ only, then similar results hold. Actually, Lemma 3-4 and Theorem 1-2 hold without change, we need only to replace $\mu^*$ by $\mu$ and the upper integral by integral, respectively in the proofs.

We can also prove the analogue of Tóth’s Theorem (Theorem 5) for $\mathcal{F}_m$:

**Theorem 5’.** $\mathcal{U} \cap \mathcal{F}_m$ is residual in $(\mathcal{F}_m, d)$.

**Proof.** See Lemma 2 in [TZs]. The only difference is in proving the density of $\mathcal{U}_0 = \{ \Phi \in \mathcal{F}_m; \Phi \text{ satisfies } (6) \text{ for some } t_0 \in \mathbb{R} \} \subset (\mathcal{F}_m, d)$, more precisely in proving that $\Psi = \Phi \chi_M + \frac{\varepsilon}{2} \chi_{\mathbb{R}\setminus M} \in \mathcal{F}_m$, where $\Phi \in \mathcal{F}_m, \varepsilon > 0$ and $M = \{ t \in \mathbb{R}; \text{ either } t \notin (0, 1) \text{ or } t \in (0, 1) \text{ and } |\Phi(t)| \geq \frac{\varepsilon}{4} \}$. 
To show this pick $f \in \mathcal{M}$, $c \in \mathbb{R}$ arbitrarily and observe that
\[
(\Psi \circ f)^{-1}([c, +\infty)) = \begin{cases} 
(\Phi \circ f)^{-1}([c, +\infty)) & \text{if } c > \frac{\varepsilon}{4} \\
(\Phi \circ f)^{-1}([c, +\infty)) \cup (f^{-1}((0, 1)) \cap (\Phi \circ f)^{-1}((-\frac{\varepsilon}{4}, \frac{\varepsilon}{4})) & \text{if } c \leq \frac{\varepsilon}{4}
\end{cases}
\]
thus $\Psi \circ f \in \mathcal{M}$. □

References


